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R. Chandrasekaran and
R. W. Shephard

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF CALIFORNIA - BERKELEY

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R. Chandrasekaran, Research Assistant
Ronald W. Shephard, Professor of Engineering Science
Department of Industrial Engineering and Operations Research
University of California, Berkeley

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1. Introduction

In simple terms there are two ways of fighting wildland fires: (i) directly attacking the combustion with water, borate bombs, or other extinguishers, (ii) indirectly constructing firebreaks which consists essentially in removing fuel along certain paths of some width which circumscribe the fire so as to stop further spread. The first method is effective for early suppression of the development of a wildland fire, while the second is generally required for well developed fires of large magnitude. We are concerned here with the optimal construction of firebreaks.

Aside from the greater importance of constructing firebreaks since they apply to larger fires, the model of this method of fire fighting lends itself more easily to an evaluation and interpretation of the parameters involved, particularly when some of them are random variables, as compared with a model for direct suppression. Further, it is easier to describe free burning fire spread models for the study of optimal firebreaks.

2. Simplifying Assumptions

First we shall assume that the fuel for wildland fires is distributed uniformly on a flat plane. This assumption is obviously an idealization of the real situation, but it may be possible to develop, in an approximate way, projections of wildlands on a plane so that equal distances in the direction of the climatological mean wind vector imply equal travel times for a free burning fire, allowing for the greater rate of travel of a fire up slopes as against the velocity of spread down slopes and also for variations in fuel distribution. Another way of interpreting this assumption in the real situation is to regard the idealized velocity of fire spread in the flat plane as an average velocity for the actual wildlands. In any event one cannot take account of every tree, bush, stream, rock, and slope.

Next, regarding fire spread models in the plane for free burning fires, the fire front in still air is assumed to propagate radially with constant velocity as illustrated in Figure 1. If a mean wind vector is superimposed, the movement of the fire front is assumed to be modified by the addition of a fixed velocity vector for the center of the fire in the direction of the mean wind, resulting in a fire front spread as illustrated in Figure 2. Generally, the rate of propagation in still air is small relative to the wind effect and the contours of fire spread in the presence of wind will be long and narrow, suggesting an approximation in the form of a plane wave front moving with constant velocity in a channel of fixed width L as illustrated in Figure 3.

The details of analysis presented in this paper will be restricted to the deterministic plane wave fire spread model of Figure 3.

A study of optimal strategies for firebreaks related to the fire spread models of both Figures 2 and 3, with wind treated as a random variable, will be reported in a doctoral thesis of one of the authors of this paper, to be published later. There a periodic review policy is used for optimal construction of firebreaks, using dynamic programming to determine the sequential alteration of the optimal firebreak paths for arbitrary sequence of realized values of the randomly varying wind.

3. Method of Constructing Firebreaks

Men with hand tools and portable equipment are used to construct a firebreak. A total crew of N men is split into n equal groups and each of these groups constructs one nth of the firebreak. For n even or odd, the firebreak sections are laid out symmetrically in a continuous path as illustrated in Figures 4 and 5. Starting at points P a section of the firebreak is constructed to a point Q. Thus the fire may be subdivided into n contiguous plane wave fires, and we need only study for given total crew size N and number of construction groups n the optimal firebreak path for a plane wave fire of width $\frac{L}{n}$. With these optimal paths,

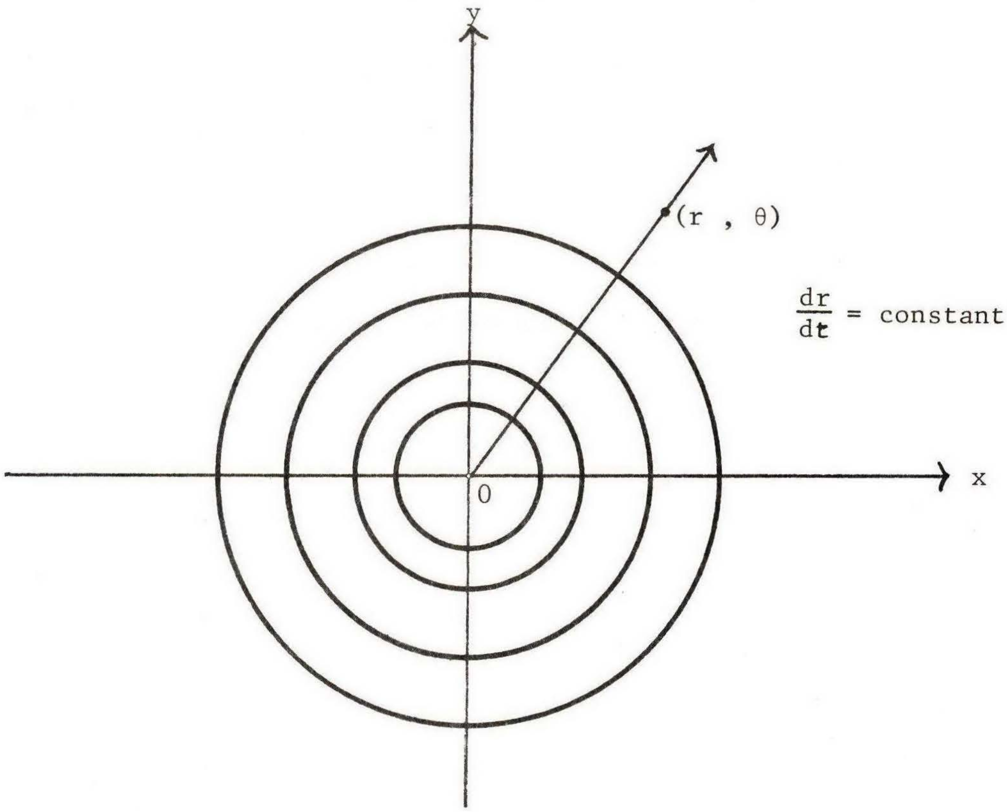

FIGURE 1. FIRE SPREAD IN STILL AIR

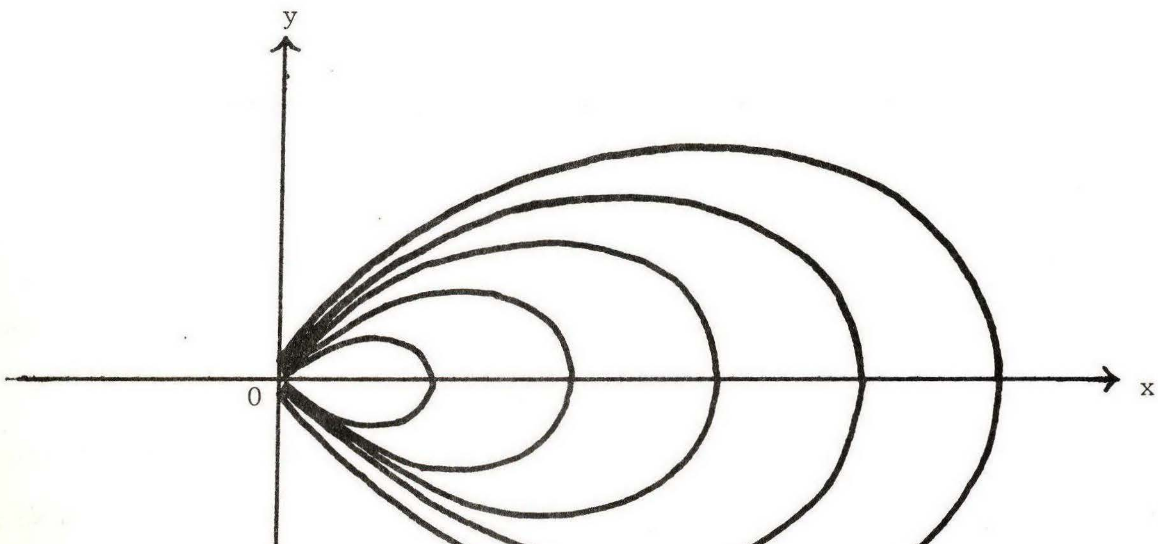

FIGURE 2. FIRE SPREAD MODIFIED BY WIND VECTOR

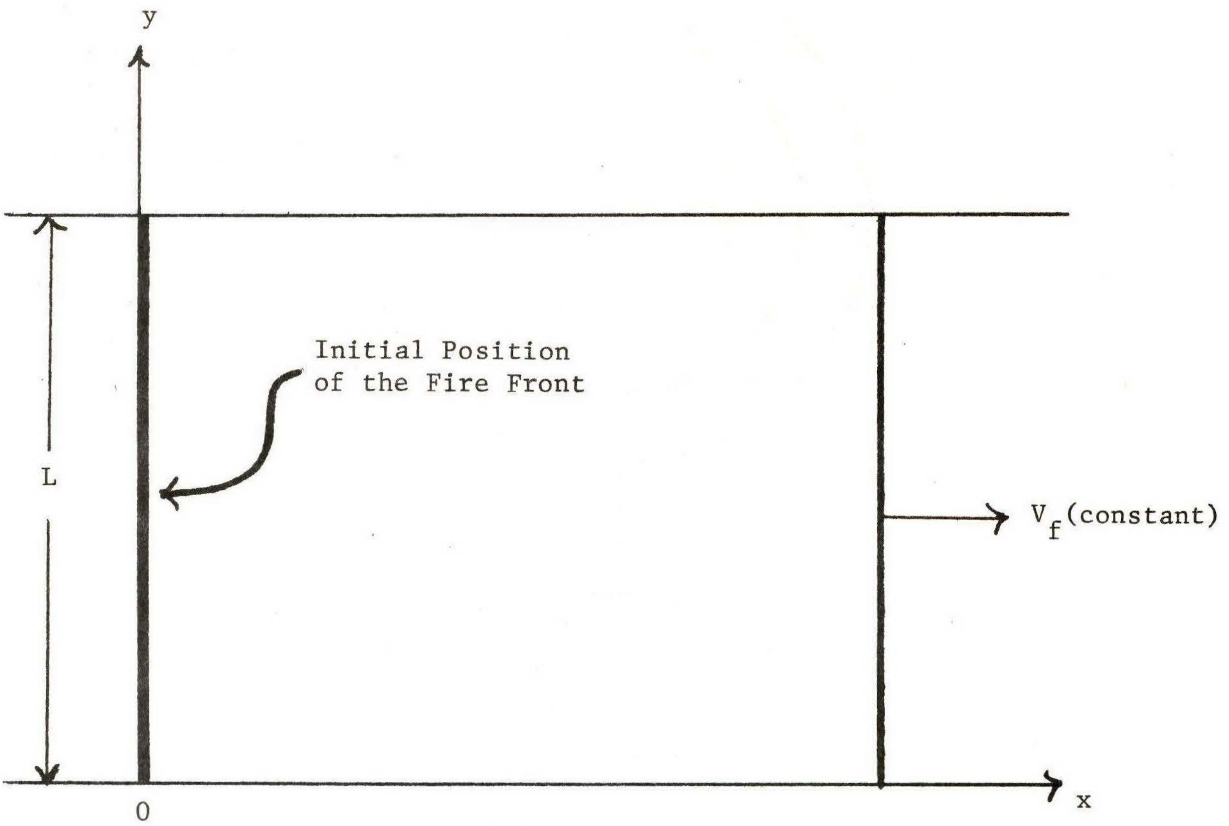

FIGURE 3. PLANE WAVE FIRE SPREAD

depending upon N and n, one may determine crew size and group size so as to minimize the resulting total costs. For this analysis it is assumed that the time required to distribute the fire fighters along a working line is negligible.

4. Cost Structure

The cost structure proposed by Parks and Jewell [4] is used. It includes the following four types of cost:

(i) A fixed cost associated with maintaining a fire fighting organization and setting it into action at the time of a fire, denoted by C_F (in dollars).

(ii) A cost proportional to the crew size N used, including items such as transportation, and other "one-shot" logistic support costs, denoted by $C_S \cdot N$ where C_S is dollars per man.

(iii) A cost proportional to the total number of man hours used in constructing firebreaks, denoted by $C_m \cdot N \cdot T_c$ where C_m is dollars per man hour and T_c is time of control.

(iv) A cost measuring the fire damage which is proportional to the area burnt, denoted by $C_B \cdot A$ where C_B is dollars per acre and A is the total area burnt.

In these terms the total cost K for any fire attacked by a crew of size N is given by

$$K = C_F + C_S \cdot N + C_m \cdot N \cdot T_c + C_B \cdot A \quad . \tag{1}$$

Although not explicity indicated in (1), the total cost depends upon the path chosen for the firebreak and the number of construction groups n, via the quantities T_c and A. In order to see this we need to consider details on extremal firebreak paths for a plane wave fire of width $\frac{L}{n}$.

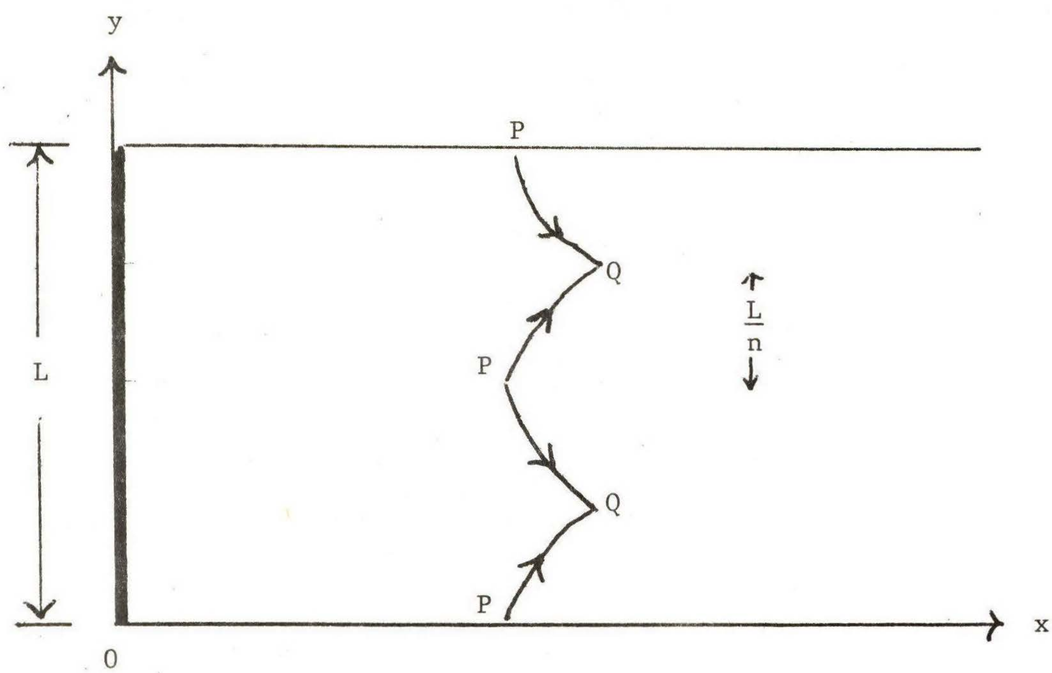

FIGURE 4. SECTIONALIZED FIREBREAK FOR n = 4

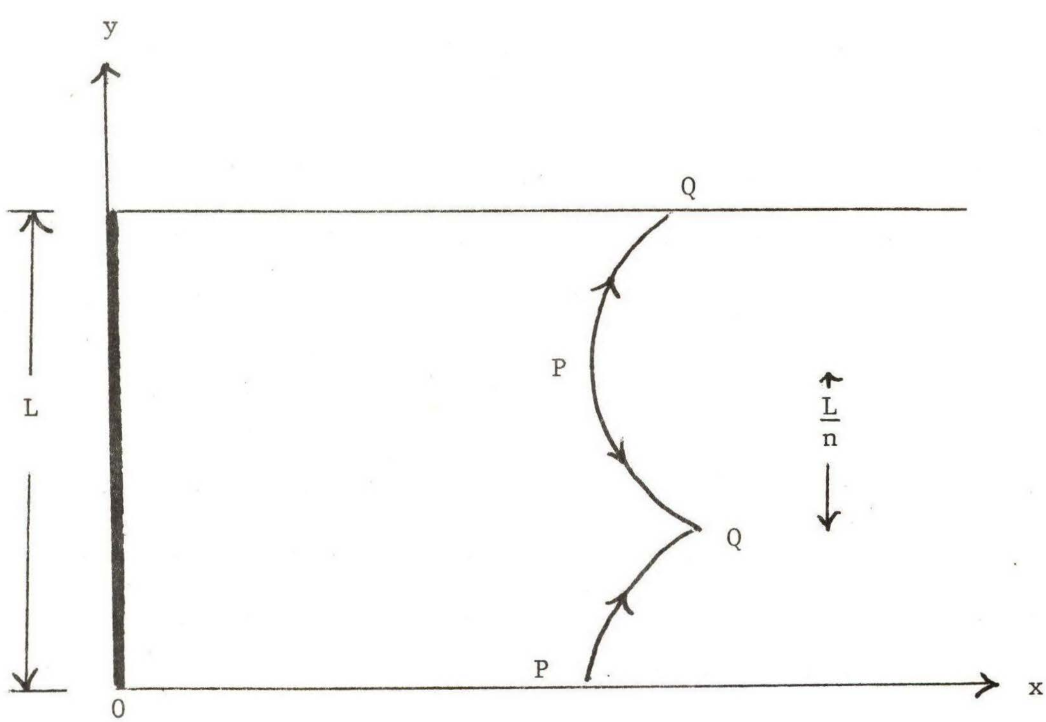

FIGURE 5. SECTIONALIZED FIREBREAK FOR n = 3

5. Extremal Firebreak Paths (Given N And n)

Define the firebreak path by a real, nonnegative single valued function $x_m = f(y)$, denoting the distance of the firebreak from the initial position of the fire front for all values of y in the interval $[0, \frac{L}{n}]$. In particular $f(0) = x_0 \geqq 0$ (See Figure 6).

For some nominal width W of the firebreak, let Δ denote the firebreak area constructed per man per unit time. Then the ratio Δ/W determines the velocity V_m of firebreak construction per man used.

The position of the fire front at any time t is given by

$$x_f(t) = V_f \cdot t \tag{2}$$

where V_f denotes the constant velocity of movement of the fire front and the origin of time is chosen so that $t = 0$ corresponds to the initial position of the fire front, i.e., $x_f(0) = 0$.

Denote by $T(y)$ the time at which the fire construction group reaches the point $(f(y), y)$. Then

$$T(y) = \frac{1}{V_m\left(\frac{N}{n}\right)} \int_0^y [1 + (f'(u))^2]^{\frac{1}{2}} \cdot du \tag{3}$$

and the time of control $T_c = T\left(\frac{L}{n}\right)$ is given by

$$T_c = \frac{1}{V_m\left(\frac{N}{n}\right)} \int_0^{L/n} [1 + (f'(u))^2]^{\frac{1}{2}} \cdot du \quad .$$

Thus, the cost function (1) depends upon the firebreak path $f(y)$ and the number of construction groups n as well as the total crew size N. Similarly, the last term of (1) depends upon n and an integral of $f(y)$.

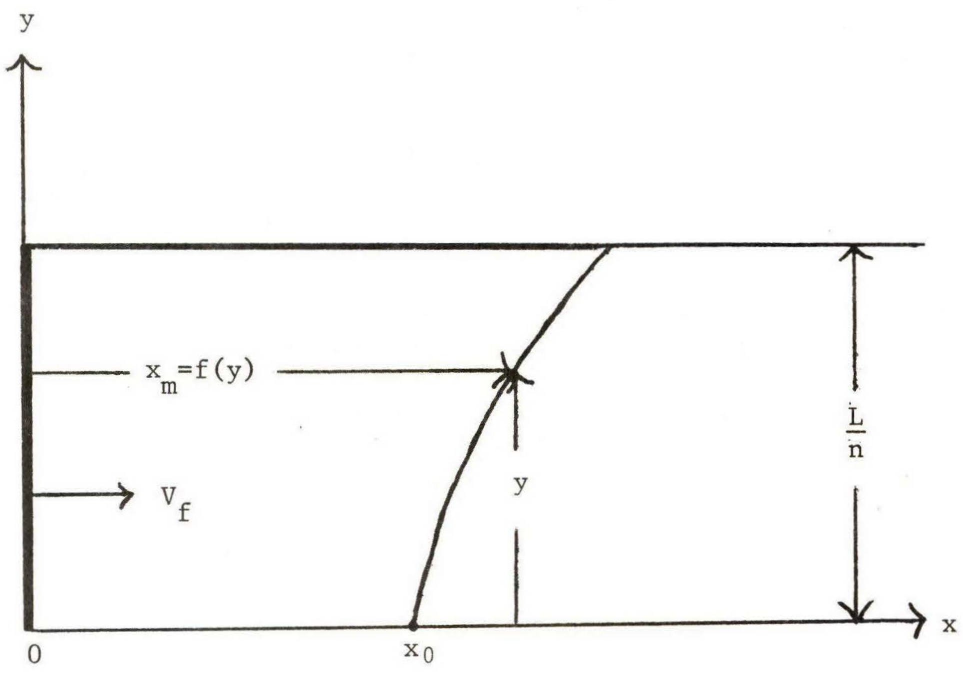

FIGURE 6. GEOMETRY OF PLANE WAVE FIREBREAK

At any time t during the construction of the firebreak, the fire front cannot pass over the firebreak path $f(y)$. Hence admissible firebreak paths $f(y)$ are restricted to satisfy the constraints,

$$(4) \qquad f(y) \geqq \frac{V_f}{V_m\left(\frac{N}{n}\right)} \int_0^y [1 + (f'(u))^2]^{\frac{1}{2}} \cdot du \quad \forall \; y \in [0 , \frac{L}{n}] \; .$$

In these variables the total cost function (1) becomes

$$(5) \qquad \begin{aligned} K[f(y) \; ; n , N] = C_F + C_S \cdot N + \frac{C_m \cdot N}{V_m\left(\frac{N}{n}\right)} \int_0^{L/n} [1 + (f'(y))^2]^{\frac{1}{2}} \cdot dy \\ + C_B \cdot n \int_0^{L/n} f(y) dy \end{aligned}$$

and extremal firebreaks are defined by those paths $f(y)$ satisfying (4) which yield

$$(6) \qquad \underset{f(y)}{\text{Min}} \; K[f(y) \; ; N , n]$$

for given N, n.

6. The Optimization Procedure

For arbitrary positive crew size N and number of construction groups n, extremal firebreak paths are determined by the constrained variational problem (6), in which the endpoints are variable on the boundaries of the plane wave fire front. In this minimization, the extremal firebreak paths may either follow the fire front, be removed from the fire front, or be a mixture of these two possibilities.

Having determined the optimal firebreak path as a function of N and n, this function is substituted into (5) to obtain the total cost as a function of

N and n , and the resulting cost function is then minimized with respect to the number of construction groups and total crew size.

Thus an optimal strategy is one which minimizes cost relative to crew size, number of groups, and firebreak path.

7. Optimal Firebreak for Given N and n

It is convenient to state certain general propositions concerning the extremal firebreak paths before undertaking the variational problem (6) .

Proposition 1: An extremal firebreak path may follow the fire front if and only if

$$\frac{V_f}{V_m\left(\frac{N}{n}\right)} < 1 \quad .$$

Proposition 2: If $\frac{V_f}{V_m\left(\frac{N}{n}\right)} < 1$ and the initial endpoint of the extremal firebreak path satisfies $f(0) = x_0 = 0$, then the extremal path entirely follows the fire front. If $\frac{V_f}{V_m\left(\frac{N}{n}\right)} \geqq 1$, there is no feasible control with this endpoint stipulation.

Corollary: An extremal firebreak path is a straight line follow-the-fire front path starting from any value of y where fire front and firebreak meet on the extremal path.

These two propositions are required as preliminaries, because the standard variational procedures do not apply for a straight line follow-the-fire front path which permits only one sided variations.

With these preliminaries, there are evidently two parameter situations to be considered for extremal firebreak paths,

$$\text{(a)} \quad \rho = \frac{V_f}{V_m\left(\frac{N}{n}\right)} \geqq 1$$

$$\text{(b)} \quad \rho = \frac{V_f}{V_m\left(\frac{N}{n}\right)} < 1$$

$$\boxed{\text{Case (a):} \quad \rho \geqq 1}$$

Here, following the fire front is not admissible, which suggests that the constraints (4) may be simplified to

$$\text{(7)} \qquad \int_0^{L/n} \left\{ \rho \sqrt{1 + f'(y)^2} - f'(y) - \frac{f(0)}{L/n} \right\} dy = 0 \quad .$$

In fact, define

$$\text{(8)} \qquad G(y) = f(y) - \rho \int_0^y \sqrt{1 + f'(r)^2}\, dr$$

and

$$\text{(9)} \qquad G'(y) = f'(y) - \rho \sqrt{1 + f'(y)^2} \quad .$$

If $\rho \geqq 1$, $G'(y) < 0$ for all $y \in [0 , \frac{L}{n}]$ and the separation between men and fire front is a strictly decreasing function of y . Thus, condition (7) is a sufficient equality replacing (4) , which is also obviously satisfied when $f(y)$ is optimal.

Hence in this case we may formulate the variational problem as follows: Define

$$\text{(10)} \qquad J(f(y)) = \int_{y_0}^{y_1} \left\{ L(y , f(y) , f'(y)) - \frac{U_0 \, f(y_0)}{y_1 - y_0} \right\} dy$$

where $y_0 = 0$, $y_1 = \frac{L}{n}$, U_0 is a multiplier and

$$L(y,f(y),f'(y)) = \frac{C_m n}{V_m}\sqrt{1+f'(y)^2} + C_B n f(y) + U_0[\rho\sqrt{1+f'(y)^2} - f'(y)] \tag{11}$$

Let $J(f(y))$ have an extremum for $f(y)$ and consider variations of $f(y)$ defined by $f(y) + h(y)$, where it is assumed that $f(y)$ and $h(y)$ are continuous and differentiable in $[0 , \frac{L}{n}]$. As boundary conditions we apply $\delta x_1 = h(y_1)$, $\delta x_0 = h(x_0)$, allowing the endpoints to vary on the lines $y = y_0$, $y = y_1$ as illustrated in Figure 7. For arbitrary $h(y)$ the increment in $J(f(y))$ is given by

$$\Delta J(f(y)) = \int_{y_0}^{y_1}\left\{L(y,f(y) + h(y),f'(y) + h'(y)) - \frac{U_0}{(y_1-y_0)}[f(y_0) + \delta x_0)]\right\}dy$$

$$- \int_{y_0}^{y_1}\left\{L(y,f(y),f'(y)) - \frac{U_0}{(y_1-y_0)}f(y_0)\right\}dy$$

and the corresponding variation δJ is

$$\delta J = \int_{y_0}^{y_1}\left\{L_{f(y)} - \frac{d}{dy} L_{f'(y)}\right\}h(y)dy + L_{f'(y)}\Big|_{y=y_1}\cdot\delta x_1$$

$$- \left[L_{f'(y)}\Big|_{y=y_0} + U_0 \right]\delta x_0$$

where $h(y)$, δx_1 and δx_0 are arbitrary. Hence, as necessary conditions for $f(y)$ to minimize $J(f(y))$ we have

$$L_{f(y)} - \frac{d}{dy} L_{f'(y)} = 0 \tag{12}$$

$$L_{f'(y)}\Big|_{y=y_0} + U_0 = 0 \tag{13}$$

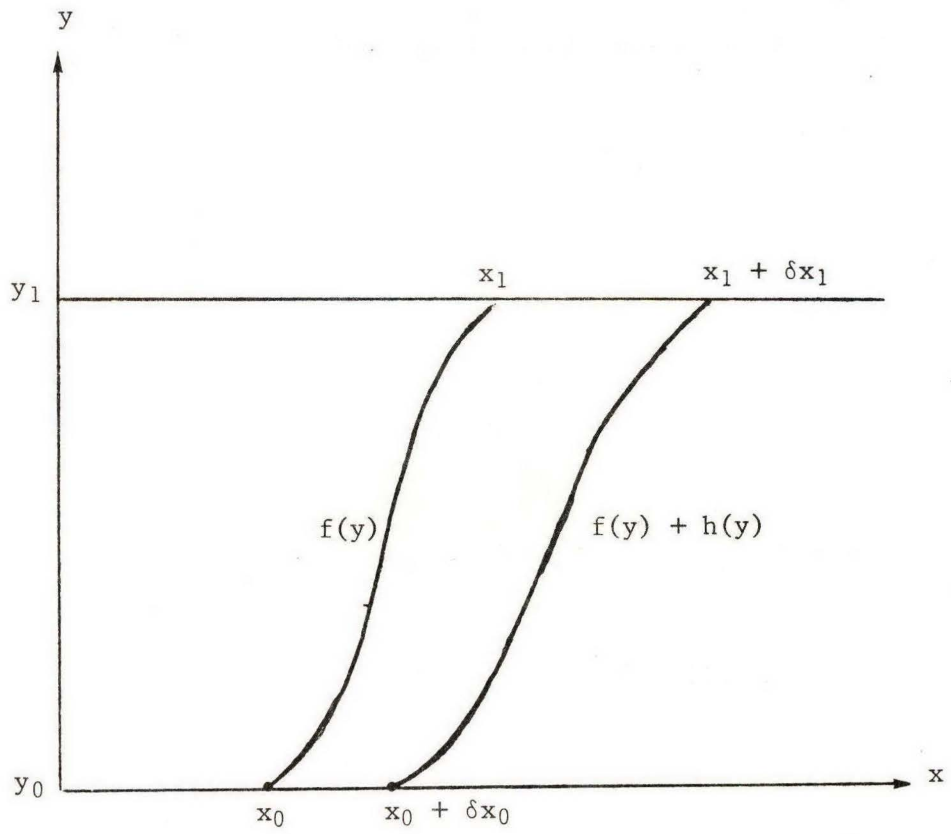

FIGURE 7. FIREBREAK VARIATIONS ($\rho \geqq 1$)

$$L_{f'(y)}\Big|_{y=y_1} = 0 \tag{14}$$

Equation (12) is the standard Euler equation, while (13) and (14) are transversality conditions for the variable endpoints.

Now, using the definition of $L(y\ ,\ f(y)\ ,\ f'(y))$, we obtain from (12)

$$\frac{f'(y)}{\sqrt{1 + f'(y)^2}} = \alpha(U_0)\cdot y + \beta \tag{12.1}$$

where β is a constant of integration and

$$\alpha(U_0) = \frac{C_B}{\frac{C_m}{V_m} + \frac{\rho}{n} U_0} \tag{15}$$

Equation (13) becomes

$$\frac{f'(0)}{\sqrt{1 + f'(0)^2}} = 0 \Rightarrow f'(0) = 0 \tag{13.1}$$

and $\beta = 0$. Equation (14) states

$$\left(\frac{C_m n}{V_m} + \rho U_0\right) \frac{f'\left(\frac{L}{n}\right)}{\sqrt{1 + f'\,\frac{L}{n}^{2}}} = U_0 \tag{14.1}$$

and using (12.1) with $\beta = 0$ one obtains

$$U_0 = C_B\cdot L \tag{16}$$

and

$$\alpha(U_0) = \alpha_0 = \frac{C_B}{\frac{C_m}{V_m} + \rho C_B \frac{L}{n}} \tag{17}$$

Therefore the extremal firebreak path for case (a) is given by

$$x_m = f(y) = x_0 + \frac{1}{\alpha_0}\left\{1 - \sqrt{1 - (\alpha_0 y)^2}\right\} \tag{18}$$

where $x_0 = f(0)$. This last equation may be written

$$[x_m - (x_0 + \frac{1}{\alpha_0})]^2 + y^2 = \frac{1}{\alpha_0^2}$$

to show that the optimal firebreak path for given n , N is a circular arc with center $(x_0 + \frac{1}{\alpha_0}, 0)$ and radius $\frac{1}{\alpha_0}$, as illustrated in Figure 8.

The initial coordinate x_0 is determined, from substitution of (18) into (7) , to be

$$x_0 = \frac{\rho}{\alpha_0}\sin^{-1}\left(\frac{\alpha_0 L}{n}\right) + \frac{1}{\alpha_0}\left[\sqrt{1 - \left(\frac{\alpha_0 L}{n}\right)^2} - 1\right] \tag{19}$$

The quantity $\frac{\alpha_0 L}{n}$ is less than unity, since $\rho \geqq 1$ for case (a) and $\alpha_0 y < 1$ for all $y \in [0, \frac{L}{n}]$. Thus we have a complete solution for the extremal firebreak path.

One final remark: the cost function (5) is convex in $f(y)$ and $f'(y)$, and the necessary conditions (12) , (13) , (14) are sufficient for the determination of the optimal firebreak path for any given positive values of N and n .

Case (b): $\rho < 1$

In this case it is possible for the firebreak to follow the fire front starting from any value of $y \in [0, \frac{L}{n}]$ (see Proposition 2 and the corollary following) and, in order to investigate the composite paths which may occur, let

$$\tilde{y} = \text{Min } y \ni f(y) = \rho \int_0^y \sqrt{1 + f'(r)^2}\,dr \tag{20}$$

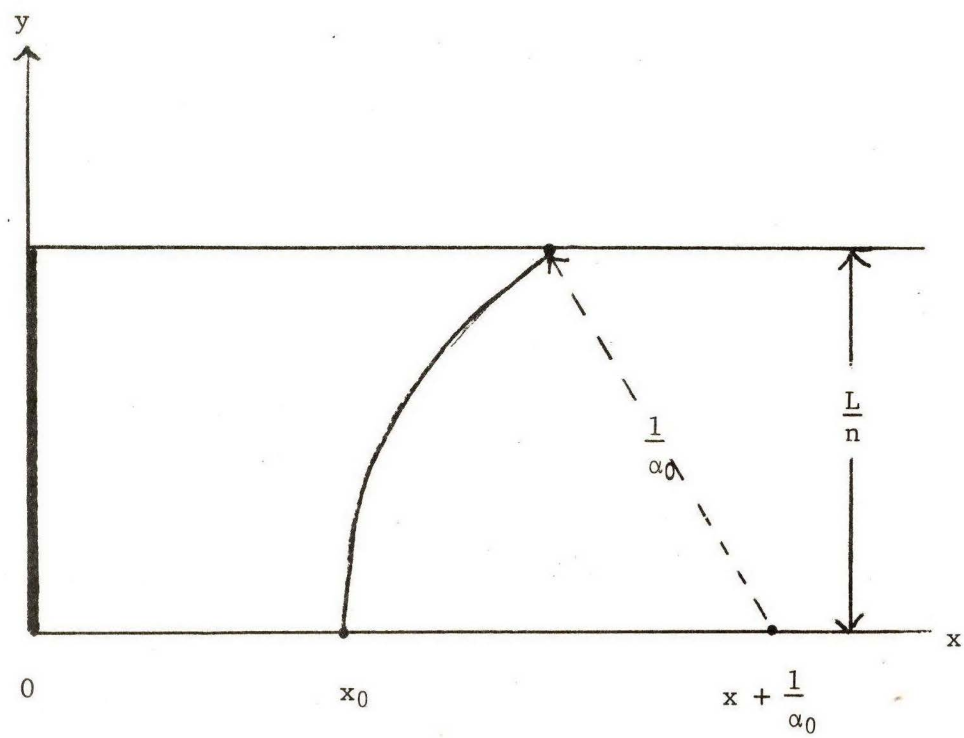

FIGURE 8. EXTREMAL FIREBREAK ($\rho \geqq 1$)

Then for $\tilde{y} \varepsilon [0 , \frac{L}{n}]$ the variational problem for extremal firebreaks over the interval $[0 , \tilde{y}]$ is one which minimizes (5) with $\left(\frac{L}{n}\right)$ in the upper integral limits replaced by $\tilde{y}$. As in case (a) we shall replace the constraints (4) by

$$\int_0^{\tilde{y}} \left\{ \rho \sqrt{1 + f'(y)^2} - f'(y) + \frac{f(0)}{\tilde{y}} \right\} = 0 \tag{7.1}$$

and show that the solution satisfies (4) for all values of $y \varepsilon [0 , \tilde{y}]$. The complete path of the optimal firebreak consists of this solution and a straight line with initial point $(f(\tilde{y}) , \tilde{y})$ and slope $\rho/\sqrt{1 - \rho^2} = \frac{dx}{dy}$.

Hence for case (b) we use the definition (11) for $L(y , f(y) , f'(y))$ and formulate the variational problem as follows: Define

$$\begin{aligned} J(f(y)) = &\int_{y_0}^{\tilde{y}} \left\{ L(y , f(y) , f'(y)) - \frac{U_0 f(y_0)}{(\tilde{y}-y_0)} \right\} dy \\ &+ \frac{C_m n}{V_m} \frac{(y_1 - \tilde{y})}{\sqrt{1 - \rho^2}} + C_B n \left\{ \tilde{x}(y_1 - \tilde{y}) + \tfrac{1}{2}(y_1 - \tilde{y})^2 \cdot \frac{\rho}{\sqrt{1-\rho^2}} \right\} \end{aligned} \tag{21}$$

taking a firebreak path $f(y)$ for $y \varepsilon [0 , \tilde{y}]$ and a straight line follow-the-fire-front path for $y \varepsilon [\tilde{y} , y_1]$, where $y_0 = 0$, $y_1 = \frac{L}{n}$, $\tilde{y} \varepsilon [0 , \frac{L}{n}]$ and U_0 is a multiplier. Let $J(f(y))$ have an extremum for $f(y)$ $(y \varepsilon [0 , \tilde{y}])$ and consider variations of $f(y)$ defined by $f(y) + h(y)$, where it is assumed that $f(y)$ and $h(y)$ are continuous and differentiable in $[0 , \tilde{y}]$. The boundary conditions are

$$\delta x_0 = h(y_0)$$

$$\delta\tilde{x} - \frac{\rho}{\sqrt{1 - \rho^2}} \cdot \delta\tilde{y} = h(\tilde{y}) ,$$

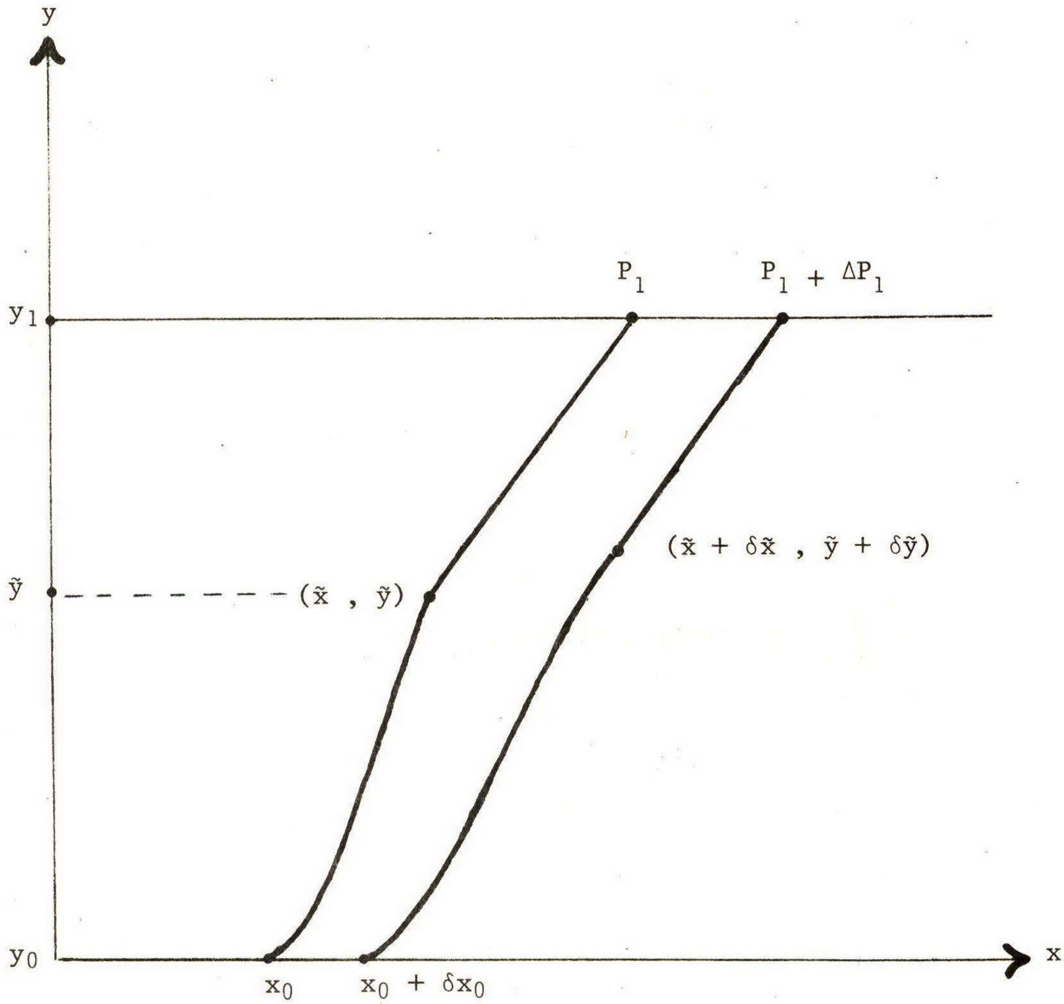

FIGURE 9. FIREBREAK VARIATIONS ($\rho < 1$)

using a general variation $(\delta\tilde{x}\ ,\ \delta\tilde{y})$ at the endpoint $(f(\tilde{x})\ ,\ \tilde{y})$ as illustrated in Figure 9. Then for arbitrary $h(y)$ the increment in $J(f(y))$ is given by

$$\Delta J(f(y)) = \int_{y_0}^{\tilde{y}+\delta\tilde{y}} \left\{ L(y,f(y)+h(y),f'(y)+h'(y)) - \frac{U_0}{(\tilde{y}+\delta\tilde{y}-y_0)}[f(y_0)+\delta x_0] \right\} dy$$

$$- \int_{y_0}^{\tilde{y}} \left\{ L(y,f(y),f'(y)) - \frac{U_0}{(\tilde{y}-y_0)}\ f(y_0) \right\} dy$$

$$+ \frac{C_m n}{V_m} \frac{1}{\sqrt{1-\rho^2}} \left\{ (y_1-\tilde{y}-\delta\tilde{y}) - (y_1-\tilde{y}) \right\}$$

$$+ C_B n \left\{ (\tilde{x}+\delta\tilde{x})(y_1-\tilde{y}-\delta\tilde{y}) + \tfrac{1}{2}(y_1-\tilde{y}-\delta\tilde{y})^2 \cdot \frac{\rho}{\sqrt{1-\rho^2}} - \tilde{x}(y_1-\tilde{y}) - \tfrac{1}{2}(y_1-\tilde{y})^2 \cdot \frac{\rho}{\sqrt{1-\rho^2}} \right\}$$

and the corresponding variation δJ is

$$\delta J = \int_{y_0}^{\tilde{y}} \left\{ L_{f(y)} - \frac{d}{dy} L_{f'(y)} \right\} h(y)\,dy$$

$$- \left\{ L_{f'(y)} \Big|_{y=y_0} + U_0 \right\} \delta x_0 + \left\{ C_B n(y_1-\tilde{y}) + L_{f'(y)} \Big|_{y=\tilde{y}} \right\} \delta\tilde{x}$$

$$+ \left\{ L \Big|_{y=\tilde{y}} - \frac{C_m n}{V_m} \frac{1}{\sqrt{1-\rho^2}} - C_B n\left(\tilde{x}+(y_1-\tilde{y})\frac{\rho}{\sqrt{1-\rho^2}}\right) - L_{f'(y)} \Big|_{y=\tilde{y}} \frac{\rho}{\sqrt{1-\rho^2}} \right\} \delta\tilde{y}$$

where $h(y)$, δx_0 , $\delta\tilde{x}$ and $\delta\tilde{y}$ are arbitrary. This problem is one of Bolza type [1] . Necessary conditions for $f(y)$ to minimize $J(f(y))$ are:

(22) $$L_{f(y)} - \frac{d}{dy} L_{f'(y)} = 0$$

(23) $$L_{f'(y)} \Big|_{y=y_0} + U_0 = 0$$

(24) $$L_{f'(y)}\Big|_{y=\tilde{y}} + C_B n(y_1-\tilde{y}) = 0$$

(25) $$L\Big|_{y=\tilde{y}} - \frac{C_m n}{V_m}\,\frac{1}{\sqrt{1-\rho^2}} - C_B n\tilde{x} = 0$$

Here, equations (23) and (24) are transversality conditions for the variable endpoints and (25) is a corner condition on the endpoint with general variation.

Now, using the definition of $L(y\ ,\ f(y)\ ;\ f'(y))$ we obtain, as before in case (a),

(22.1) $$\frac{f'(y)}{\sqrt{1+f'(y)^2}} = \alpha(U_0)y + \beta$$

Equation (23) has the same form as (13.1) and implies $\beta = 0$. Equation (24) , although modified relative to (14) , again implies $U_0 = C_B L$ and $\alpha(U_0)$ is again defined by (17) . Hence the extremal firebreak path for the interval $[0\ ,\ \tilde{y}]$ is given by equation (18) .

The corner condition (25) yields

(26) $$\tilde{y} = \frac{\rho}{\alpha_0}$$

giving a determination of the endpoint $(f(\tilde{y})\ ,\ \tilde{y})$. However, since $\rho < 1$, equation (26) does not surely determine a value for $\tilde{y}$ which is less than $\left(\frac{L}{n}\right)$. But, using (18) for the extremal path in the cost function (5) with $\tilde{y}$ replacing $\left(\frac{L}{n}\right)$, the resulting expression is convex in $\tilde{y}$ for any positive N and n . Hence the solution for $\tilde{y}$ may be written

(27) $$\tilde{y} = \begin{cases} \dfrac{\rho}{\alpha_0} & \text{if } \rho < \dfrac{\alpha_0 L}{n} \text{ or } n^2 < \left(\alpha_0 L\dfrac{V_m}{V_f}\right)N \\[2ex] \dfrac{L}{n} & \text{if } \rho \geqq \dfrac{\alpha_0 L}{n} \text{ or } n^2 \geqq \left(\alpha_0 L\dfrac{V_m}{V_f}\right)N \end{cases}$$

The initial coordinate x_0 for case (b) is determined (by using (18), (27) and (7.1)) to be

$$x_0 = \frac{\rho}{\alpha_0} \sin^{-1}(\rho) + \frac{1}{\alpha_0}\left[\sqrt{1 - \rho^2} - 1\right] \tag{28}$$

when $n^2 < \left(\alpha_0 L \frac{V_m}{V_f}\right) N$, otherwise it is given by equation (19).

The question remains - does this solution satisfy the constraints (4). Using (18) in (9), we obtain

$$G'(y) = \frac{\alpha_0 (y - \frac{\rho}{\alpha_0})}{\sqrt{1 - (\alpha_0 y)^2}}$$

and, since $y < \tilde{y} \leqq \frac{\rho}{\alpha_0}$ for all $y \in [0, \tilde{y})$, $G(y)$ is strictly decreasing in the interval $[0 ; \tilde{y}]$ to a zero value at $\tilde{y}$. Thus the constraints (4) are satisfied. Moreover $\alpha_0 y \leqq \rho < 1$ for all $y \in [0, \tilde{y}]$.

The general form of the optimal firebreak is illustrated in Figure 10, consisting of a circular arc connected to a straight line follow-the-fire-front piece with slope $\rho/\sqrt{1 - \rho^2}$, and at $y = \tilde{y} < \frac{L}{n}$ the slope $f'(\tilde{y})$ may be computed from (18) and (27) to have the same value. Hence the condition (25) is in effect a Weierstrass-Erdman corner condition [2].

Summary for Cases (a) and (b)

In Case (a), recall that $\rho \geqq 1$ and $\frac{\alpha_0 L}{n} < 1$. Hence it follows that

$$\rho \geqq 1 \Rightarrow n \geqq \frac{V_m N}{V_f}$$

$$\frac{\alpha_0 L}{n} < 1 \Rightarrow n > \alpha_0 L$$

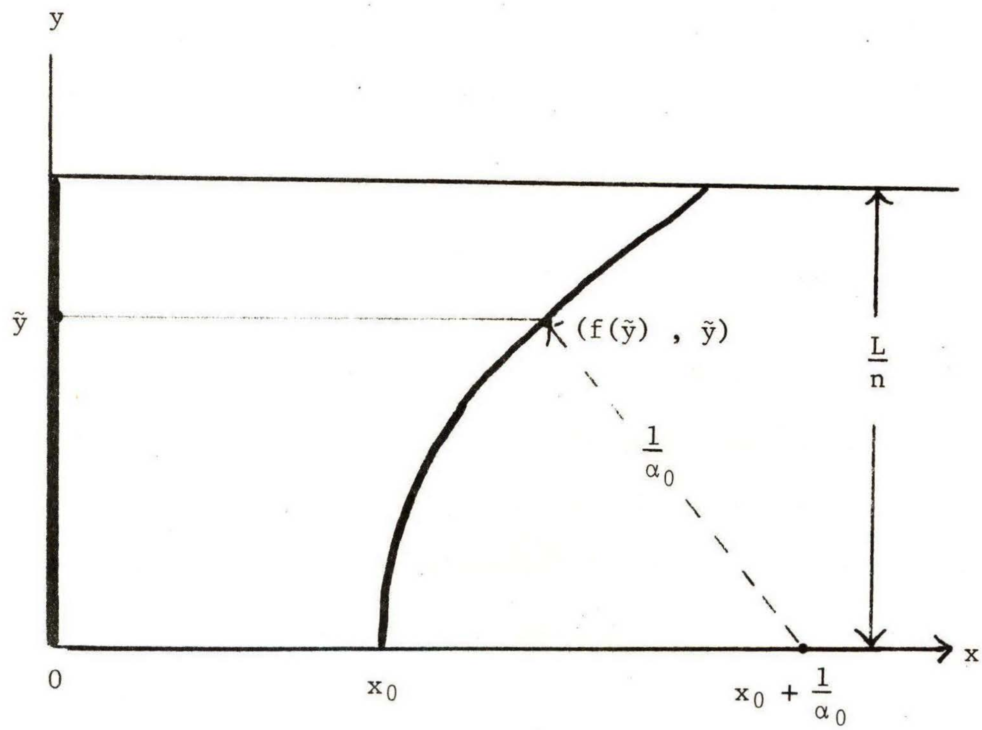

FIGURE 10. EXTREMAL FIREBREAK ($\rho < 1$)

ARBITRARY CORNER $\tilde{y}$

and

$$n^2 \geqq \alpha_0 L \frac{V_m N}{V_f} \quad .$$

Thus, if we regard $\tilde{y}$ to be a variable for Case (a) as well as Case (b), we may state for both cases that

$$(29) \qquad \tilde{y} = \begin{cases} \dfrac{\rho}{\alpha_0} & \text{if } n^2 < \alpha_0 L \dfrac{V_m N}{V_f} \\[2ex] \dfrac{L}{n} & \text{if } n^2 \geqq \alpha_0 L \dfrac{V_m N}{V_f} \end{cases}$$

and the optimal firebreak path for given (N , n) is

$$(30) \qquad f(y) = \begin{cases} x_0 + \dfrac{1}{\alpha_0}\left\{1 - \sqrt{1 - (\alpha_0 y)^2}\right\} & \text{for } 0 \leqq y \leqq \tilde{y} \\[2ex] x_0 + \dfrac{1}{\alpha_0}\left\{1 - \sqrt{1 - \rho^2}\right\} + \dfrac{\rho}{\sqrt{1 - \rho^2}} \cdot y & \text{for } \tilde{y} \leqq y \leqq \dfrac{L}{n} \end{cases}$$

where $\rho = \dfrac{V_f}{V_m\left(\frac{N}{n}\right)}$ and

$$(31) \qquad \left.\begin{aligned} x_0 &= \frac{\rho}{\alpha_0} \sin^{-1}(\alpha_0 \tilde{y}) + \frac{1}{\alpha_0}\left\{\sqrt{1 - (\alpha_0 \tilde{y})^2} - 1\right\} \\[2ex] \alpha_0 &= \frac{C_B}{\dfrac{C_m}{V_m} + \dfrac{C_B L V_f}{V_m N}} \end{aligned}\right\}$$

8. Optimal Number of Construction Groups and Optimal Crew Size

For arbitrarily given crew size N , the total cost under optimal firebreak path depends upon the number of construction groups. Treating n continuously,

we investigate the cost function $K(n\ ;\ N)$ and seek the optimal number of construction groups n^* for arbitrary $N > 0$.

The function $K(n\ ;\ N)$ has two forms depending upon whether

$$\text{Case (i)} \qquad n^2 \geqq \alpha_0 L\left(\frac{V_m N}{V_f}\right)$$

$$\text{Case (ii)} \qquad n^2 < \alpha_0 L\left(\frac{V_m N}{V_f}\right)$$

In the first case, it follows that,

$$K(n\ ;\ N) = C_F + C_S\cdot N + \frac{C_B L}{2\alpha_0}\sqrt{1 - \left(\frac{\alpha_0 L}{n}\right)^2} + \frac{C_B n}{2(\alpha_0)^2}\sin^{-1}\left(\frac{\alpha_0 L}{n}\right) \tag{32}$$

and

$$\frac{\partial K}{\partial n} = \frac{C_B}{2(\alpha_0)^2}\left\{\sin^{-1}\left(\frac{\alpha_0 L}{n}\right) - \left(\frac{\alpha_0 L}{n}\right)\sqrt{1 - \left(\frac{\alpha_0 L}{n}\right)^2}\right\} > 0 \tag{33}$$

for all $n^2 \geqq \alpha_0 L\left(\frac{V_m N}{V_f}\right)$, since this inequality implies $\frac{\alpha_0 L}{n} < 1$ when $\rho < 1$ and $\frac{\alpha_0 L}{n} < 1$ for $\rho \geqq 1$ (see Case (a) above). Hence $K(n\ ;\ N)$ is a monotone increasing function of n for all N in the range $n^2 \geqq \alpha_0 L\left(\frac{V_m N}{V_f}\right)$.

In the second case, it follows that

$$K(n\ ;\ N) = C_F + C_S\cdot N + \left(\frac{C_m L}{V_m} + \frac{C_B L^2}{2n}\rho\right)\frac{1}{\sqrt{1 - \rho^2}} - \frac{C_B n}{2(\alpha_0)^2}\left[\frac{\rho}{\sqrt{1 - \rho^2}} - \sin^{-1}(\rho)\right] \tag{34}$$

and

$$\frac{\partial K}{\partial n} = \frac{C_B}{2(\alpha_0)^2}\sin^{-1}(\rho) - \frac{1}{2C_B}\left(\frac{C_m}{V_m}\right)^2 \frac{\rho}{(1-\rho^2)^{3/2}} \tag{35}$$

$$\frac{\partial^2 K}{\partial n^2} = \frac{\rho}{n\sqrt{1-\rho^2}}\left[\frac{C_B}{2(\alpha_0)^2} - \frac{1}{2C_B}\left(\frac{C_m}{V_m}\right)^2 \frac{(1+2\rho^2)}{(1-\rho^2)^2}\right] \tag{36}$$

for all $0 < n^2 < \alpha_0 L\frac{V_m N}{V_f}$. Although (34) is not monotone, it can be shown that, since $\rho < 1$ in this case, $\frac{\partial^2 K}{\partial n^2} < 0$ whenever $\frac{\partial K}{\partial n} = 0$ and $\frac{\partial K}{\partial n}_{n=0+} = 0$. Thus over the range $\left(0 < n^2 < \alpha_0 L\frac{V_m N}{V_f}\right)$ the function $K(n\ ;\ N)$ is quasi-concave. Further $K(n\ ;\ N)$ is continuous at $n^2 = \alpha_0 L\frac{V_m N}{V_f}$ having a form as illustrated in Figure 11, and clearly the optimal number of construction groups (not restricted to integers) for given N is

$$n^* = \sqrt{\alpha_0 L\frac{V_m N}{V_f}}\ . \tag{37}$$

Then, except for integer disparity, it follows from (29) that the optimal fire-break path is entirely an arc of a circle.

Turning now to the optimization of crew size, the cost function $K(N)$ resulting from (37) is

$$\begin{aligned} K(N) &= C_F + C_S\cdot N + \frac{C_B L}{2\alpha_0}\sqrt{1-\frac{\alpha_0 L V_f}{V_m N}} \\ &\cdot \frac{C_B\sqrt{\alpha_0 L\frac{V_m N}{V_f}}}{2(\alpha_0)^2}\sin^{-1}\left(\sqrt{\frac{\alpha_0 L V_f}{V_m N}}\right) \end{aligned} \tag{38}$$

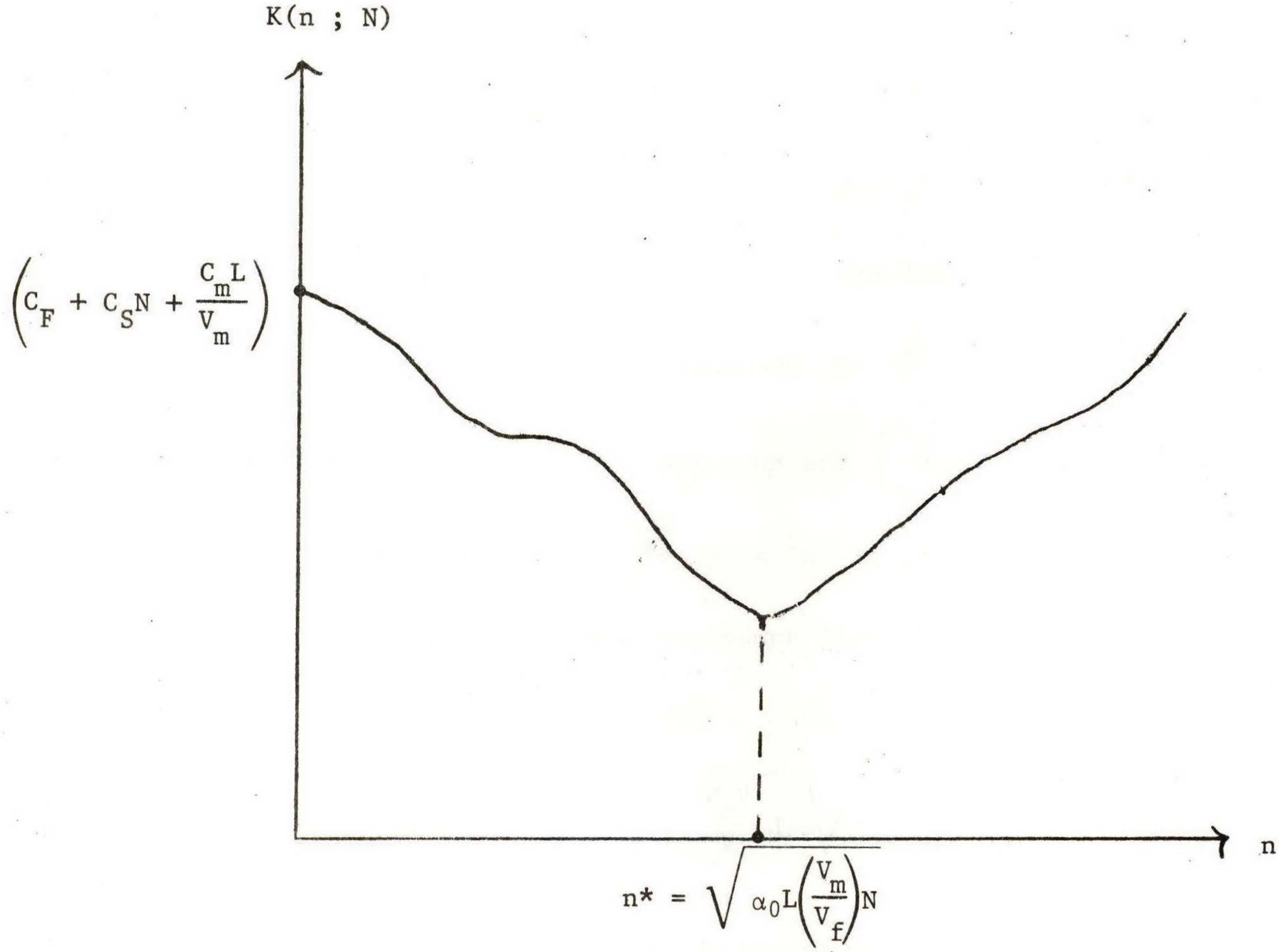

FIGURE 11. TOTAL COST AS FUNCTION OF NUMBER OF CONSTRUCTION GROUPS FOR OPTIMAL FIREBREAK PATH AND ARBITRARY CREW SIZE

where α_0 depends upon N as given by (17) . It is convenient to make a change of variable.

$$\frac{1}{Q} = \frac{\alpha_0 L V_f}{V_m N} = \frac{C_B L}{\frac{C_m}{V_f} N + C_B L} \tag{39}$$

Then the cost function $K(N)$ is written

$$\begin{aligned} K(Q) &= C_F + \frac{C_S}{C_m} V_f C_B L(Q - 1) \\ &+ \frac{L}{2}\left(\frac{C_m}{V_m}\right)\left[\sqrt{\frac{Q}{Q - 1}} + \frac{Q^{3/2}}{(Q - 1)} \sin^{-1}\left(\frac{1}{\sqrt{Q}}\right)\right] \end{aligned} \tag{40}$$

where Q is a monotone increasing function of N and $1 < Q < \infty$ as $0 < N < \infty$. Further

$$K'(Q) = \frac{C_S}{C_m} V_f C_B L - \frac{1}{4}\left(\frac{C_m}{V_m}\right)\left\{\frac{Q + 1}{\sqrt{Q}(Q - 1)^{3/2}} + \frac{\sqrt{Q}(3 - Q)}{(Q - 1)^2} \sin^{-1}\left(\frac{1}{\sqrt{Q}}\right)\right\} \tag{41}$$

Evidently $K(Q)$ has a minimum for some finite value $Q^0 > 1$, since $K(Q)$ is positive over the range $1 < Q < \infty$ with $K'(Q) \to -\infty$ for $Q \to 1$ and $K'(Q) > 0$ for $Q \to \infty$.

Hence, treating N continuously in the range $0 < N < \infty$, the minimum of $K(N)$ is obtained at a value

$$N^0 = (Q^0 - 1) C_B L\left(\frac{V_f}{C_m}\right) \tag{42}$$

and, if the maximum available number of men is $\bar{N}$ the optimal crew size is given by

$$N^* = \begin{cases} N^0 & \text{if } N^0 \leqq \bar{N} \\ \bar{N} & \text{if } N^0 > \bar{N} \end{cases} . \tag{43}$$

With this optimal value of N, the optimal number of construction groups n^* is determined from (37) and in turn the optimal firebreak path is computed from equations (29), (30), and (31).

Note that this solution always yields a circular arc firebreak meeting the fire front during construction only at the endpoint $y = \frac{L}{n}$. But, if we restrict N and n to be integers such that $\left(\frac{N}{n}\right)$ is an integer, the optimal firebreak may have a terminal straight line portion following the fire front.

In order to obtain the integral optimal values for crew size and number of construction groups, define

$$[a] = \text{largest integer equal to or less than } a$$

$$\langle a\rangle = \text{smallest integer equal to or greater than } a .$$

Then compare the following four costs:

$$\text{(i)} \quad K\left([n^*],\ [n^*]\cdot\left[\frac{N^*}{n^*}\right]\right)$$

$$\text{(ii)} \quad K\left([n^*],\ [n^*]\cdot\left\langle\frac{N^*}{n^*}\right\rangle\right)$$

$$\text{(iii)} \quad K\left(\langle n^*\rangle,\ \langle n^*\rangle\cdot\left[\frac{N^*}{n^*}\right]\right)$$

$$\text{(iv)} \quad K\left(\langle n^*\rangle,\ \langle n^*\rangle\cdot\left\langle\frac{N^*}{n^*}\right\rangle\right)$$

and select the corresponding integral optimal values for n and N.

Notice that the following subproblems which may be of interest in themselves have also been solved:

(1) Given N and n, finding an optimal firebreak path.

(2) Given N, finding the optimal number of construction groups and the optimal firebreak path for each.

(3) Given that $N \leqq \bar{N}$, finding the optimal values of N and n as well as the optimal path for the firebreaks.

It is interesting to observe that certain intuitive results for particular cases are obtained by substituting the relevant values of the parameters.

For example, when $C_S = 0$ (i.e., there are no transportation or "one-shot" logistic costs) and unlimited men are available, we obtain $N^* = \infty$, $n^* = \infty$, $x_0 = 0$ and the optimal firebreak consists of a straight line which coincides with the initial position of the fire, recalling that we assumed the time required to get the fire fighters distributed along a working line was negligible. Moreover the related optimized total cost is finite, being

$$C_F + L \frac{C_m}{V_m}$$

obtained by computing $\lim_{Q\to\infty} K(Q)$ from (40) .

Next consider $C_m = 0$, i.e., a situation where fire fighting is done by volunteers. Then, it follows from equations (33) and (35) that $n^* = 1$ for any N , and from (40) it follows that $N^* = 1$. Then from (29)

$$\tilde{y} = \begin{cases} \left(\dfrac{V_f}{V_m}\right)^2 & \text{if} \quad V_f < V_m \\ L & \text{if} \quad V_f \geq V_m \end{cases}$$

and optimal firebreak paths of both types arise depending upon the relative size of V_f and V_m .

If $C_B = 0$, it is clear from the structure of the problem that $N^* = 0$ implying that it is optimal not to fight the fire.

9. Stochastic Models and Related Optimization

So far we have considered only a deterministic model for the study of an optimal strategy. The most significant random variable to be allowed for is the

velocity of fire spread V_f . Taking V_f as a nonnegative random variable with known probability distribution, it is reasonable on practical grounds to assume a finite upper bound on the realized values of this random variable. Then an optimal strategy is one which minimizes expected cost.

For this extension a multi-stage dynamic programming decision model may be used, in which the dispatching of firebreak construction units is done periodically based on observed values of the velocity of fire spread at predetermined points of time.

As in the deterministic model, a number of construction groups n is used with an assumption that each one nth of the fire front has the same realized value of fire velocity V_f at the beginning of any review period. The value of n is taken fixed for all periods in order to determine the optimal policy for any number of construction groups, and the resulting suboptimized cost function $K(n)$ may be minimized to determine the optimal number of construction groups n^* .

For arbitrary n , the optimal policy is a feedback control rule which minimizes the total expected cost for an unknown number of periods, in which an equal number of men are dispatched to each construction group at the beginning of each review period based upon the realized values of fire velocity in all previous periods. In practice the optimal firebreak path during any period is determined in accordance with the realized value of fire velocity at the beginning of the period, and with the terminal point of the firebreak for the previous period as the initial point for the firebreak in the current period. The policy structure is open end, i.e., no fixed number of review period are set in advance, and the terminating stage is reached when the men enclose the fire front. For the extremal problem, the main difference relative to the deterministic case is that the variational problem has a fixed initial point for all periods except the first. The initial point of the firebreak in the first period is a free endpoint of the related variational problem and it is determined by the optimization of cost with

respect to n .

If it arises that the fire front overtakes the fire line, then the policy operates to dispatch additional men to each construction group so as to construct a follow-the-fire front firebreak. This involves a cost of dispatching in addition to the operating cost of the extra men, and the optimal policy is determined under this rule of dispatching. Since the transportation cost C_S is incurred the instant a man is sent and the total cost is otherwise a decreasing function of the construction group size, it is never optimal to call men back until completion of the firebreak.

REFERENCES

[1] Bliss, G. A., Lectures on the Calculus of Variations, University of Chicago Press, Chapter VII, (1959).

[2] Bolza, O., Lectures on the Calculus of Variations, G. E. Stechart and Company, New York.

[3] Parks, George M., "Analytical Models for Attack and Control of Wildland Fires," ORC 65-8, Operations Research Center, University of California, Berkeley, (April 1964).

[4] Parks, George M., and W. S. Jewell, "A Preliminary Model for Initial Attack," Research Report IER-172-31, Operations Research Center, University of California, Berkeley.